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Transmission of water waves through small apertures

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A method of solution is proposed for flows through small apertures in otherwise impermeable barriers. This method, which is an application of the method of matched asymptotic expansions, is used to solve a specific water-wave problem, yielding an approximate formula for the transmission coefficient.

1. Introduction

The specific problem to be studied here concerns the transmission of small amplitude water waves through a narrow horizontal slit in a vertical wall. The fluid is taken to be infinitely deep and inviscid, the wall is infinitesimally thin and impermeable apart from the slit, and the free surface condition is linearized. In particular, no account is taken of real-fluid effects at the sharp edges of the slit.

An exact 'solution' of the above classical two-dimensional water-wave problem is in principle available from the work of Lewin (1963) (see also Mei 1966). These treatments reduce problems involving thin vertical barriers of a general nature to Hilbert problems, as discussed at length by Muskhelishvili (1946). However, specialization of the exact formulae to this particular case of a single slit in an infinite wall presents formidable analytical and computational problems and no comparisons have yet been made with the simple approximate results to be presented here.

The approximation to be made here is that the width of the slit is small compared with the other significant length scales of the problem, namely the depth of submersion of the slit and the wavelength of the incident waves. The method of matched asymptotic expansions (Van Dyke 1964) is used, in an abbreviated semi-intuitive manner, to construct inner and outer approximations to the flow, from which the transmission coefficient is readily obtained. In fact the main reason for presenting this work is to illustrate this method of approach, although there may be practical applications (especially of some extended work to be suggested later) to breakwater or other engineering problems.

Some essentials of the present approach are contained in Rayleigh's (e.g. 1897) work on diffraction of sound waves through small apertures (see also Lamb, 1932, p. 532). It is interesting to note that Rayleigh was already thinking in such terms last century, even though he did not use the formal language or apparatus of 'matched asymptotic expansions'.

For instance, in his 1897 paper, Rayleigh used the idea that, in the neighbourhood of the aperture, the flow is purely local without far-field influence apart from the over-all scale of velocities. Indeed, in the acoustic case even the field equation changes in the inner region from the wave equation to Laplace's equation. This inner solution then 'matches' an outer solution which looks just like an acoustic source situated at the position of the (infinitesimal) aperture. A similar understanding of the flow situation is essential to the present work, with the addition of some important points of detail.

One of these points of detail concerns the fact that although the inner *flow* appears not to interact with the outer flow, the same cannot be said for the inner velocity *potential*. Thus the inner potential (both in the water-wave and acoustic contexts) is a solution of Laplace's equation for flow through an aperture in a wall of infinite extent. This flow must be source-like at a great distance to the right of the wall, and sink-like at a great distance to the left. But if (for example) the left-ward limit of the velocity potential is

$$\phi \to \log r \tag{1.1}$$

(r being distance from the aperture), then the right-ward limit is *not*

$$\phi \to -\log r,$$

but rather

$$\phi \to -\log r + c, \tag{1.2}$$

where c is a *unique* constant for the given aperture geometry. In fact c is equal to $2\log\frac{1}{2}a$ in the present case where $2a$ is the slit width, but more generally one may consider c to be a known or easily computed property of the aperture geometry in cases where the aperture is other than a sharp-edged slit in an infinitesimally thin wall.

When matching of inner and outer solutions takes place, the influence of this constant c is felt directly in the outer region, and its value determines the value of the transmission coefficient. It might be considered surprising that an additive constant in a velocity potential should ever be significant, since the potential is generally used only in differentiated form. However, in the present unsteady flow, there is an effect via the free-surface boundary condition (2.1), which involves time as well as space derivatives of ϕ. Newman (1969) has found it necessary to use a 'constant' analogous to c in solving a slender-body cross-flow problem; in his case the effect is felt because his c can depend on the co-ordinate normal to the cross-flow plane. On the other hand, such an effect would seem to be absent from the acoustic problems treated by Rayleigh (1897).

2. Formulation of the problem

The flow situation is as sketched in figure 1. Fluid velocity is expressed as the gradient of a potential $\phi(x,y,t)$ satisfying Laplace's equation in $y<0$ and the linearized free-surface condition.

$$g\,\partial\phi/\partial y + \partial^2\phi/\partial t^2 = 0, \tag{2.1}$$

on $y=0$. There is no flow across a wall occupying most of the plane $x=0$, i.e.

$$\partial\phi/\partial x = 0 \quad \text{on} \quad x=0, \quad |y+h|>a, \tag{2.2}$$

where h is the depth of the centre of the slit and $2a$ its width.

The boundary conditions at infinity state that there is an incident wave of fixed amplitude from the left, together with outgoing waves (reflected to the left, transmitted to the right) whose magnitudes are definite (determinable) fractions of the incident amplitude. In mathematical terms, we suppose that as $x \to -\infty$,

$$\phi(x,y,t) \to \mathscr{R}[A_I \exp(-i\kappa z + i\sigma t) + \rho A_I \exp(-i\kappa z - i\sigma t)], \tag{2.3}$$

and as $x \to +\infty$,

$$\phi(x,y,t) \to \mathscr{R}[\tau A_I \exp(-i\kappa z + i\sigma t)], \tag{2.4}$$

where $z = x+iy$, $\kappa = \sigma^2/g$, A_I is an arbitrary (fixed) complex constant, and ρ, τ are complex constants to be determined. It is common to specify $A_I \equiv 1$, and this can be done without loss of generality; however, we shall make a different choice.

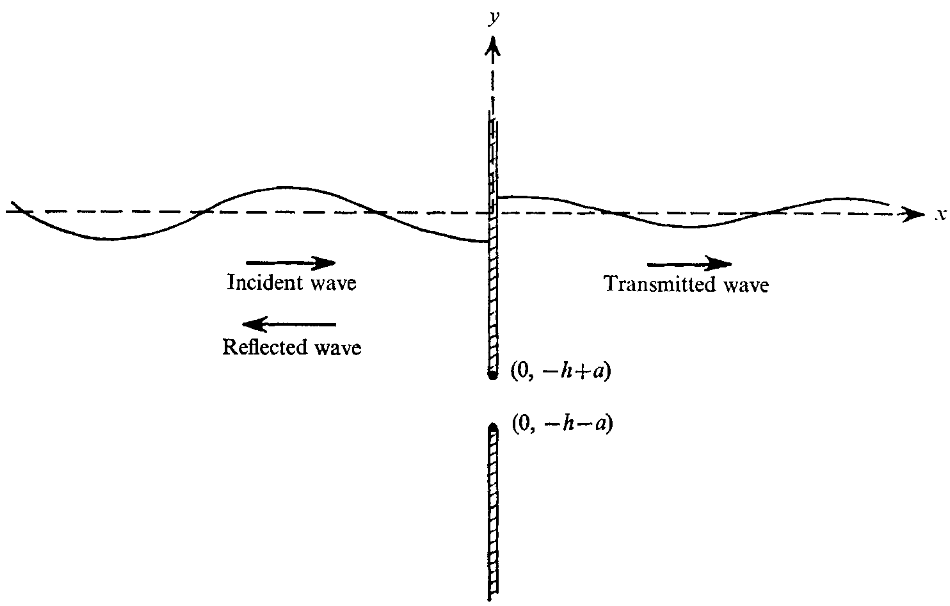

FIGURE 1

The quantities ρ and τ are of course the reflexion and transmission coefficients in complex form. Consistent with the notation in (2.3) and (2.4) we shall adopt an analytic complex potential $f(z,t)$, where $\phi = \mathscr{R}f$.

3. The outer problem

Clearly if the slit is narrow, i.e. $2a/h$ is small, the effect of the slit, as seen by an observer on the right side of the wall, will be as if the slit is a source of fluid of oscillating strength under a free surface. At the same time an observer on the left side will see a sink (i.e. source of opposite strength), the magnitude of the singularity strength being the same, by continuity. Giving mathematical expression to this idea, we write on the right side $x \geqslant 0$,

$$f = F(z,t), \tag{3.1}$$

where

$$F(z,t) = \left[\frac{1}{2\pi}\log\frac{z+ih}{z-ih} - \frac{1}{\pi}\int_0^\infty \frac{\exp(-ik(z-ih))}{k-\kappa}\,dk\right]\cos\sigma t$$
$$-\exp(-i\kappa(z-ih))\sin\sigma t \qquad (3.2)$$

(Wehausen & Laitone 1960, p. 481; the integral takes its Cauchy principal value). In choosing this particular form we have made an arbitrary assumption about the magnitude (unity) and phase of the oscillating source strength. This is entirely equivalent to an *a priori* choice of the *transmitted* amplitude τA_I in (2.4), and costs us nothing in the generality of the final result. Specifically, we observe that, as $x \to \pm\infty$,

$$F(z,t) \to \pm i \exp(-i\kappa(z-ih) \pm i\sigma t), \qquad (3.3)$$

so that, in the notation of (2.4), we have chosen

$$\tau A_I = ie^{-\kappa h}. \qquad (3.4)$$

On the other hand, the flow on the left side $x \leqslant 0$ will be represented by

$$f = -F(z,t) + (A\cos\sigma t + B\sin\sigma t)\exp(-i\kappa(z-ih)), \qquad (3.5)$$

where A and B are real constants, to be determined. Note that $F(z,t)$ itself is defined for *all* x, and represents outgoing waves on both sides of the wall, on account of the asymptotic representation (3.3). Furthermore, it generates a flow which is symmetric in x and hence satisfies (2.2) for all $y \neq -h$. The representation (3.5) contains a term $-F$, which gives the correct singular behaviour at $z = -ih$, but represents only outgoing (i.e. to the left) waves. The extra terms A, B, constitute standing waves which satisfy (2.2) if A, B are real, and which can be split into incident and reflected contributions.

Thus, as $x \to -\infty$, we have

$$f(z,t) \to \left(\frac{A}{2} + \frac{B}{2i}\right)\exp(-i\kappa(z-ih) + i\sigma t) + \left(i + \frac{A}{2} - \frac{B}{2i}\right)\exp(-i\kappa(z-ih) - i\sigma t). \quad (3.6)$$

Again, in the notation of (2.3), we identify

$$A_I = \left(\frac{A}{2} + \frac{B}{2i}\right)e^{-\kappa h}, \qquad (3.7)$$

$$\rho A_I = \left(i + \frac{A}{2} - \frac{B}{2i}\right)e^{-\kappa h}, \qquad (3.8)$$

so that the reflexion coefficient is

$$\rho = \left(i + \frac{A}{2} - \frac{B}{2i}\right)\Big/\left(\frac{A}{2} + \frac{B}{2i}\right), \qquad (3.9)$$

and from (3.4) the transmission coefficient is

$$\tau = i\Big/\left(\frac{A}{2} + \frac{B}{2i}\right). \qquad (3.10)$$

The problem is solved if we can find A, B, but these constants can not be determined from the outer problem alone. To determine A, B, we must look closer into the details of the flow near the slit. In order to prepare the ground for this, let us first look at the behaviour of the outer solution near the (relatively!) infinitesimal aperture at $z = -ih$. This behaviour is immediate from the definitions, and we have

$$f(z,t) \to \left(\frac{\cos\sigma t}{2\pi}\right)\log(z+ih)+C_+(t) \tag{3.11}$$

as $z \to 0_+ - ih$, and

$$f(z,t) \to -\left(\frac{\cos\sigma t}{2\pi}\right)\log(z+ih)+C_-(t) \tag{3.12}$$

as $z \to 0_- - ih$, where

$$C_+(t) = \left[-\frac{1}{2\pi}\log(-2ih)-\frac{1}{\pi}\int_0^\infty \frac{e^{-2kh}}{k-\kappa}dk\right]\cos\sigma t - e^{-2\kappa h}\sin\sigma t \tag{3.13}$$

and

$$C_-(t) = -C_+(t)+e^{-2\kappa h}(A\cos\sigma t+B\sin\sigma t). \tag{3.14}$$

Equations (3.11), (3.12), merely express the fact that, near the aperture, the solution looks like a simple source-sink pair in an infinite fluid, of oscillating strength $\cos\sigma t$. The expressions C_+, C_- are additive 'constants' (with respect to space) which carry within them into the inner problem the whole of the wave-like nature of the outer problem – hence the apparent complexity of the defining equation (3.13).

4. The inner problem

Since the slit is narrow and is submerged to a distance h large compared with its width $2a$, the flow in the immediate neighbourhood of the slit will be as if the free surface were not present. In fact it will be exactly the usual potential flow through a finite slit in an *infinite* wall, the fluid extending to infinity in *all* directions, as in figure 2.

The solution of the inner problem is readily obtained by use of a Joukowski transformation of the form

$$z+ih = -\tfrac{1}{2}ia(\zeta+\zeta^{-1}), \tag{4.1}$$

which maps the whole cut z plane onto the upper half ζ plane, with correspondence between points and regions as shown in figure 3. A solution which represents a source of strength m at the origin of the ζ plane is

$$f = (m/2\pi)\log\zeta+C, \tag{4.2}$$

where C is any constant. This solution does in fact provide a flow such as that sketched in figure 2 ($m > 0$), as we see by looking at the properties of the mapping (4.1) for both large and small $|\zeta|$.

First, as $|\zeta| \to \infty$, we obtain the flow behaviour in the neighbourhood of the points $\mathscr{C}\mathscr{B}\mathscr{J}$ in both planes, i.e. the limit $x \to +\infty$ in the z plane, or the right-hand semi-circle at infinity. But then from (4.1)

$$z+ih \to -\tfrac{1}{2}ia\zeta$$

so that

$$f \to -\frac{m}{2\pi}\log\left(\frac{z+ih}{-\frac{1}{2}ia}\right)+C = -\frac{m}{2\pi}\log(z+ih)+\left[C+\frac{m}{2\pi}\log\left(-\tfrac{1}{2}ia\right)\right], \quad (4.3)$$

a source-like flow away from the point $z = -ih$. Similarly, as $|\zeta| \to 0$, we obtain the neighbourhood of $\mathscr{EFG}$, i.e. the limit $x \to -\infty$ or the left-hand semi-circle at infinity, when

$$z+ih \to -\tfrac{1}{2}ia/\zeta$$

so that

$$f \to \frac{m}{2\pi}\log(z+ih)+\left[C-\frac{m}{2\pi}\log(-\tfrac{1}{2}ia)\right], \quad (4.4)$$

a sink-like flow toward $z = -ih$.

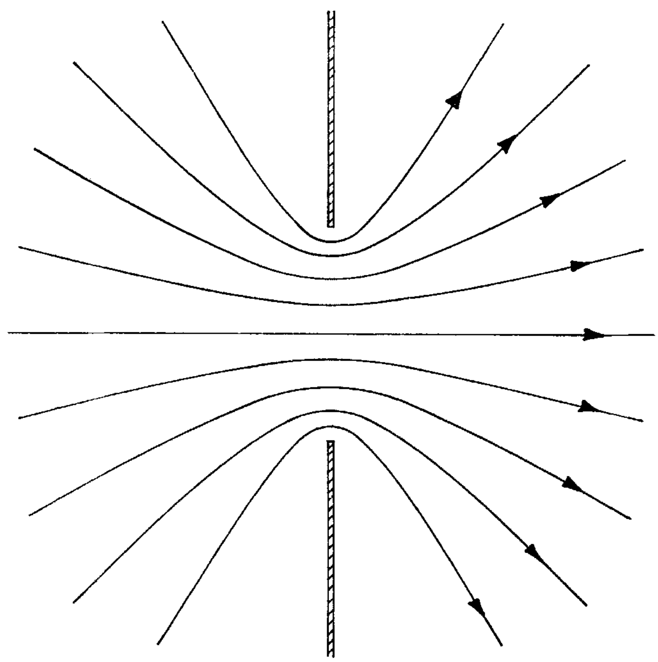

FIGURE 2

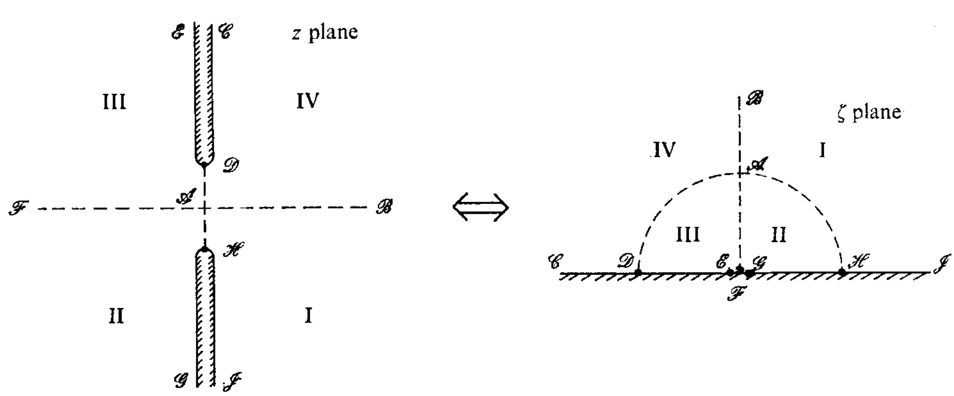

FIGURE 3

It is important to notice that the constant terms in (4.3) and (4.4) are different. While it is true to say that we can if we wish ignore any *one* additive constant (such as C) in solving a Neumann boundary-value problem, it is not true that the additive constants in two different expansions of the *same* flow are the same. For instance, we could choose $C = (m/2\pi)\log(-\frac{1}{2}ia)$ whereupon the left-side flow is just $f \to (m/2\pi)\log(z+ih)$. However, the right-side flow is *not* just $f \to -(m/2\pi)\log(z+ih)$, but needs to include the constant term $(m/\pi)\log(-\frac{1}{2}ia)$.

5. Matching

Intuitively the matching principle simply states that the inner region seen from the outside looks like the other region seen from the inside. Let us imagine a long-sighted giant of height about h, who can see only what is going on at that length scale. Then if this giant attempts to focus his eyes on the aperture near $z = -ih$ he will see only a blurred source-like picture. On the other hand, a short-sighted midget of height about a, sitting in the opening will be able to see only the detailed flow through the opening. At the extreme outer edge of his vision will *also* be a blurred source-like flow, and it is important that these two blurred pictures are identical.

In mathematical terms, equations (3.11) and (3.12) provide the inner behaviour of the outer solution, whereas (4.3) and (4.4) give the outer behaviour of the inner solution, and matching asserts that these expressions must be identical term by term. This is achieved easily enough by taking

$$m = -\cos\sigma t, \tag{5.1}$$

$$C_+ = C + (m/2\pi)\log(-\tfrac{1}{2}ia), \tag{5.2}$$

and

$$C_- = C - (m/2\pi)\log(-\tfrac{1}{2}ia). \tag{5.3}$$

Thus, on subtraction,

$$C_+ - C_- = -(\cos\sigma t/\pi)\log(-\tfrac{1}{2}ia). \tag{5.4}$$

But from (3.14),

$$C_+ + C_- = e^{-2\kappa h}(A\cos\sigma t + B\sin\sigma t). \tag{5.5}$$

Thus

$$2C_+ = e^{-2\kappa h}(A\cos\sigma t + B\sin\sigma t) - (\cos\sigma t/\pi)\log(-\tfrac{1}{2}ia). \tag{5.6}$$

Since C_+ is a known combination (3.13) of $\cos\sigma t$ and $\sin\sigma t$, equation (5.6) furnishes all the information we need to find A, B.

Specifically, on equating coefficients of $\cos\sigma t$, $\sin\sigma t$, we have

$$B = -2 \tag{5.7}$$

and

$$A e^{-2\kappa h} = -\frac{1}{\pi}\log(-2ih) - \frac{2}{\pi}\int_0^\infty \frac{e^{-2kh}}{k-\kappa}dk + \frac{1}{\pi}\log(-\tfrac{1}{2}ia);$$

i.e.

$$A = \frac{e^{2\kappa h}}{\pi}\log\left(\frac{a}{4h}\right) + \frac{2}{\pi}\overline{Ei}(2\kappa h), \tag{5.8}$$

where

$$\overline{Ei}(v) = \int_{-\infty}^{v} \frac{e^u}{u}du \tag{5.9}$$

is an exponential integral (Jahnke & Emde 1945, p. 2).

6. The transmission coefficient

Since we have now found the coefficients A, B, we can immediately write down from (3.9) and (3.10) the reflexion coefficient ρ and transmission coefficient τ. Let us concentrate on the latter, in view of its obvious interest in this particular problem as a measure of the permeability of the wall. In any case, the well-known relationship

$$|\rho|^2+|\tau|^2 = 1$$

(readily verifiable in this particular case) enables us to obtain at least the magnitude of ρ from that of τ.

Using (3.10), (5.7) and (5.8) we obtain

$$\tau = i\Big/\left(\frac{e^{2\kappa h}}{2\pi}\log\frac{a}{4h}+\frac{1}{\pi}\overline{Ei}\,(2\kappa h)+i\right). \tag{6.1}$$

The magnitude of τ is of greatest interest, and we have

$$|\tau|^{-2} = 1+\left[\frac{e^{2\kappa h}}{2\pi}\log\frac{a}{4h}+\frac{1}{\pi}\overline{Ei}(2\kappa h)\right]^2. \tag{6.2}$$

This expression can be considered to be a function of two dimensionless quantities, one the ratio $2a/h$ of width to submersion of the aperture (assumed small), and the other the ratio $h/\lambda = \kappa h/2\pi$ between depth of aperture and the wavelength $\lambda = 2\pi/\kappa$ of the incident wave. The latter ratio has been assumed neither small nor large in the analysis.

Clearly as the slit becomes vanishingly narrow, the transmission tends to zero as it should, but at a very slow (inverse logarithmic) rate. Thus for fixed κh the term in $\log(a/4h)$ eventually dominates (6.2) so that $|\tau|^{-2}\to\infty$ and $|\tau|\to 0$. On the other hand, for fixed $2a/h$, $|\tau|\to 0$ also as $\kappa h\to 0$ or ∞. Clearly as $h/\lambda\to\infty$, the exponential $e^{2\kappa h}$ and exponential integral both tend at an exponential rate to infinity, so that τ is exponentially small. This we should expect, since the slit (of fixed width) is ultimately many wavelengths submerged and sees very little of the incident energy. At the other extreme, as $h/\lambda\to 0$, the exponential integral $\overline{Ei}(2\kappa h)$ becomes logarithmically large, so that $\tau\to 0$ like the inverse of the logarithm of $2\kappa h$. Again this is a very slow rate, indicating that much of the energy of very long waves gets through even quite narrow slits.

This is illustrated by figure 4, showing graphs of $|\tau|^2$ against h/λ for various values of $2a/h$. The left side of the figure ($h/\lambda\to 0$) corresponds to *long* waves and the right side ($h/\lambda\to\infty$) to short waves, relative to the submergence h of the slit. Maximum transmission of energy occurs at a wavelength which gets longer as the slit width is decreased. At $2a/h = 0{\cdot}4$ as much as 72 % of the energy is transmitted at $h/\lambda = 0{\cdot}6$; however, it may be that this slit is too wide for the present theory to be acceptable without reservation. On the other hand, even at $2a/h = 0{\cdot}05$, which would surely qualify as a narrow slit, 40 % transmission occurs for $h/\lambda \sim 0{\cdot}02$, i.e. for a wave with a length 1000 times the slit width.

The question of whether or not such high transmission would be achieved in practice (in the presence of separation and energy dissipation at the edges of the slit, and other real-fluid effects) is beyond the scope of the present paper. It is worthy of note, however, that substantial transmission of long waves through small holes is not uncommon in classical water-wave theory. For instance, we can infer from the results of Dean (1945, p. 273) that a submerged barrier extending up to only one unit from the water surface would transmit 33 % of the energy of a wave 500 units long. Similar conclusions are obtained by Evans (1970) for a finite submerged barrier.

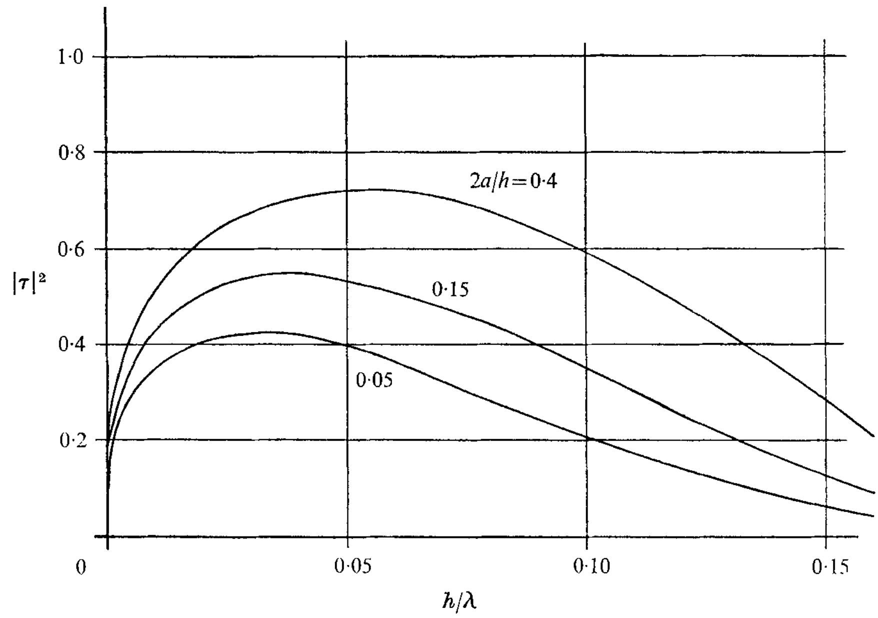

FIGURE 4

It would of course be desirable to compare figure 4 with independent estimates of the transmission coefficient, particularly with exact values calculated from the theory of Lewin (1963) and with suitable experimental measurements. Attempts are presently being made to provide both of these comparisons.

At the same time we should like to generalize the present flow geometry. In particular, it is not difficult to modify the shape of the aperture, including effects of wall thickness or rounded corners. This is particularly desirable, of course, if comparison with experiment is to be made. On the other hand, the exact theory of Lewin (1963) can apply only to infinitesimal walls, so that the generalized geometries mentioned above will lead to results which cannot be compared with any exact theory. Other less obvious generalizations include effects of finite water depth or three-dimensional holes; in these cases the outer as well as the inner approximation must be modified.

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